The LISA Response Function

Neil J. Cornish and Louis J. Rubbo
Department of Physics, Montana State University, Bozeman, MT 59717

The orbital motion of the Laser Interferometer Space Antenna (LISA) introduces modulations into the observed gravitational wave signal. These modulations can be used to determine the location and orientation of a gravitational wave source. The complete LISA response to an arbitrary gravitational wave is derived using a coordinate free approach in the transverse-traceless gauge. The general response function reduces to that found by Cutler[1] for low frequency, monochromatic plane waves. Estimates of the noise in the detector are found to be complicated by the time variation of the interferometer arm lengths.

I. INTRODUCTION

In this paper we derive the response of the Laser Interferometer Space Antenna (LISA) to an arbitrary gravitational wave. Our detector response function includes the full orbital motion and is valid at all frequencies. Previous treatments have either assumed that LISA is stationary with respect to the background sky[2, 3, 4], or have been limited to the low frequency limit[1, 5].

The LISA mission[6] calls for three spacecraft to orbit the Sun in an equilateral triangular formation. The center-of-mass for the constellation, known as the guiding center, follows a circular orbit at 1 AU and has an orbital period of one year. In addition to the bulk motion of the detector about the Sun, the triangular formation will cartwheel about the guiding center in a clockwise manner as seen by an observer at the Sun. The orbital motion introduces frequency (Doppler), amplitude, and phase modulation into the observed gravitational wave signal. These effects have been calculated in the low frequency limit, where the antenna pattern is well approximated by a quadrupole, and the Doppler modulation is due to the guiding center motion. At higher frequencies the antenna pattern becomes more complicated, and the "rolling" Doppler modulation due to the cartwheel motion has to be included.

The divide between high and low frequencies roughly coincides with the divide between gravitational waves with wavelengths shorter or longer than the arms of the detector. To be precise, the dividing line is the transfer frequency $f_* \equiv c/(2\pi L)$, where L is the unperturbed distance between spacecraft[3]. For LISA, with its mean arm length of 5×10^6 km, the transfer frequency has an approximate value of 10 mHz. Above the transfer frequency the antenna pattern is distinctly non-quadrapolar[7]. The frequencies at which the guiding center and rolling motion impart measurable effects are easily estimated. Since both the guiding center and cartwheel motions have periods of one year, the Doppler modulations will enter as sidebands separated by the modulation frequency $f_m = 1/\text{year}$. Equating the modulation frequency to the Doppler shift, $\delta f \simeq (v/c)f$, for motion with velocity v yields the characteristic frequency $f_v = cf_m/v$ at which Doppler modulation becomes measurable. Assuming 5×10^6 km armlengths the cartwheel turns with velocity $v/c = 0.192 \times 10^{-5}$, while the guiding center moves with velocity $v/c = 0.994 \times 10^{-4}$. Thus, the Doppler modulation due to the guiding center's motion becomes measurable at frequencies above $f_{gc} = 0.3$ mHz, while the rolling cartwheel motion becomes important above $f_r = 16$ mHz. This demonstrates that at low frequencies only the bulk motion of the detector needs to be considered, while at high frequencies the cartwheel motion also needs to be included.

Our calculations are performed using the coordinate free approach introduced in Ref.[4] (see also Ref. [8] for a closely related approach). This allows our results to be applied to any variation on the current LISA design, or to any follow on mission. The low frequency limit of our general detector response function yields a simple result that can be shown to agree with Cutler's[1]. Throughout this paper we use natural units where $G = c = 1$, however we will report all frequencies in terms of Hertz.

II. DETECTOR RESPONSE

The detector response to a gravitational wave source located in the $\hat{n}$ direction can be found using Barycentric coordinates $(t, \mathbf{x})$ and the transverse-traceless gauge to describe a plane gravitational wave $\mathbf{h}(q, \widehat{\Omega})$ propagating in the $\widehat{\Omega} = -\hat{n}$ direction. The surfaces of constant phase are given by $\xi = t + \hat{n} \cdot \mathbf{x} = \text{const.}$. A general gravitational wave can be decomposed into two polarization states:

$$\mathbf{h}(\xi, \hat{n}) = h_+(\xi)\mathbf{e}^+ + h_\times(\xi)\mathbf{e}^\times \tag{1}$$

where $\mathbf{e}^+$ and $\mathbf{e}^\times$ are the polarization tensors

$$\begin{aligned} \mathbf{e}^+ &= \hat{u} \otimes \hat{u} - \hat{v} \otimes \hat{v}, \\ \mathbf{e}^\times &= \hat{u} \otimes \hat{v} + \hat{v} \otimes \hat{u}. \end{aligned} \tag{2}$$

The basis vectors $\hat{u}$, $\hat{v}$ and the source location $\hat{n}$ can be expressed in terms of the location of the source (θ, ϕ) according to

$$\begin{aligned} \hat{u} &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{v} &= \sin\phi\,\hat{x} - \cos\phi\,\hat{y} \\ \hat{n} &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\,. \end{aligned} \tag{3}$$

Following the Doppler tracking calculations described in Ref.[4], we find that the optical path length between spacecraft i and spacecraft j may be written as

$$\begin{aligned}\ell_{ij}(t_i) &= \int_i^j \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \\ &= |\mathbf{x}_j(t_j) - \mathbf{x}_i(t_i)| \\ &\quad + \frac{1}{2}(\hat{r}_{ij}(t_i)\otimes\hat{r}_{ij}(t_i)) : \int_i^j \mathbf{h}(\xi(\lambda))\, d\lambda\,. \end{aligned} \tag{4}$$

Here $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $|\mathbf{x}-\mathbf{y}|$ denotes the Cartesian distance between $\mathbf{x}$ and $\mathbf{y}$. The unit vector

$$\hat{r}_{ij}(t_i) = \frac{\mathbf{x}_j(t_j) - \mathbf{x}_i(t_i)}{\ell_{ij}(t_i)} \tag{5}$$

points from spacecraft i at the time of emission, t_i, to spacecraft j at the time of reception, t_j. Finally, the quantity $\xi(\lambda)$ is the parameterized wave variable

$$\xi(\lambda) = t(\lambda) - \hat{\Omega}\cdot\mathbf{x}(\lambda)\,. \tag{6}$$

Explicitly, the time and position depend on the parameterization in the following way

$$t(\lambda) = t_i + \lambda \tag{7}$$

$$\mathbf{x}(\lambda) = \mathbf{x}_i(t_i) + \lambda\,\hat{r}_{ij}(t_i)\,. \tag{8}$$

The variation in the Cartesian distance between the spacecraft can be separated into a contribution due to the Sun and other Solar system bodies, and a small perturbation due to the gravitational wave. Denoting the unperturbed spacecraft locations by $\mathbf{x}^0(t)$ and integrating the geodesic equation for the metric $g_{\mu\nu}$ yields

$$\begin{aligned}|\mathbf{x}_j(t_j) - \mathbf{x}_i(t_i)| &= |\mathbf{x}_j^0(t_j) - \mathbf{x}_i^0(t_i)| \\ &+ \hat{r}_{ij}(t_i)\cdot\int^j \mathbf{h}(t - \hat{\Omega}\cdot\mathbf{x}_j(t))\cdot\frac{d\mathbf{x}_j(t)}{dt}\,dt \\ &- \hat{r}_{ij}(t_i)\cdot\int^i \mathbf{h}(t - \hat{\Omega}\cdot\mathbf{x}_i(t))\cdot\frac{d\mathbf{x}_i(t)}{dt}\,dt\,. \end{aligned} \tag{9}$$

The gravitational wave dependent terms in the above equation can be ignored as they are down by a factor of $v = |d\mathbf{x}(t)/dt| \simeq 10^{-4}$ compared to the leading order gravitational wave contribution described in (4). Thus we may write

$$\ell_{ij}(t_i) = \ell_{ij}^0(t) + \delta\ell_{ij}(t_i) \tag{10}$$

where

$$\ell_{ij}^0(t_i) = |\mathbf{x}_j^0(t_j) - \mathbf{x}_i^0(t_i)|\,, \tag{11}$$

and

$$\delta\ell_{ij}(t_i) = \frac{1}{2}\frac{\hat{r}_{ij}(t_i)\otimes\hat{r}_{ij}(t_i)}{1 - \hat{\Omega}\cdot\hat{r}_{ij}(t_i)} : \int_{\xi_i}^{\xi_j}\mathbf{h}(\xi)\,d\xi\,. \tag{12}$$

Here we have used

$$\frac{\mathrm{d}\xi(\lambda)}{\mathrm{d}\lambda} = 1 - \hat{\Omega}\cdot\hat{r}_{ij}(t_i) \tag{13}$$

to make a change of variables in the integration. Defining $\mathbf{H}(a,b)$ to be the antiderivative of the gravitational wave

$$\mathbf{H}(a,b) \equiv \int_b^a \mathbf{h}(\xi)\,d\xi \tag{14}$$

simplifies our expression further to

$$\delta\ell_{ij}(t_i) = \frac{1}{2}\frac{\hat{r}_{ij}(t_i)\otimes\hat{r}_{ij}(t_i)}{1 + \hat{n}\cdot\hat{r}_{ij}(t_i)} : \mathbf{H}(\xi_j, \xi_i)\,. \tag{15}$$

where we have used the relationship $\hat{\Omega} = -\hat{n}$. To leading order in $\mathbf{h}$, the time of reception t_j is defined in terms of the time of emission t_i by the implicit relation

$$\ell_{ij}^0(t_i) = |\mathbf{x}_j^0(t_i + \ell_{ij}^0(t_i)) - \mathbf{x}_i^0(t_i)|\,. \tag{16}$$

The gravitational wave can be decomposed into frequency components

$$\mathbf{h}(\xi) = \int_{-\infty}^{\infty}\tilde{\mathbf{h}}(\omega)e^{i\omega\xi}\,d\omega\,, \tag{17}$$

which allows us to write

$$\mathbf{H}(a,b) = \int_{-\infty}^{\infty}\frac{\tilde{\mathbf{h}}(\omega)}{i\omega}\left(e^{i\omega a} - e^{i\omega b}\right)d\omega \tag{18}$$

and

$$\delta\ell_{ij}(t_i) = \ell_{ij}(t_i)\int_{-\infty}^{\infty}\mathbf{D}(\omega, t_i, \hat{n}) : \tilde{\mathbf{h}}(\omega)e^{i\omega\xi_i}d\omega \tag{19}$$

where the one-arm detector tensor is given by

$$\mathbf{D}(\omega, t_i, \hat{n}) = \frac{1}{2}\left(\hat{r}_{ij}(t_i)\otimes\hat{r}_{ij}(t_i)\right)\mathcal{T}(\omega, t_i, \hat{n}) \tag{20}$$

and the transfer function takes the form

$$\begin{aligned}\mathcal{T}(\omega, t_i, \hat{n}) &= \mathrm{sinc}\left[\frac{\omega}{2\omega_{ij}}(1 + \hat{n}\cdot\hat{r}_{ij}(t_i))\right] \\ &\quad\times\ \exp\left[i\frac{\omega}{2\omega_{ij}}(1 + \hat{n}\cdot\hat{r}_{ij}(t_i))\right]\,. \end{aligned} \tag{21}$$

Here $\omega_{ij} = 1/\ell_{ij}(t_i)$ is the angular transfer frequency for the arm.

The connection between the optical path length variations and the detector output depends on the interferometer design. The original proposal was to use laser transponders at the end-stations to send back a phased locked signal. A more recent proposal is to eliminate the transponders and turn LISA into a virtual interferometer where the signal is put together in software. The raw ingredients for this procedure are the phase differences between the received and transmitted laser light along each arm. The signal transmitted from spacecraft i that is received at spacecraft j at time t_j has its phase compared to the local reference to give the output $\Phi_{ij}(t_j)$. The phase difference has contributions from the laser phase noise, $C(t)$, optical path length variations, shot noise $n^s(t)$ and acceleration noise $\mathbf{n}^a(t)$:

$$\begin{aligned}\Phi_{ij}(t_j) &= C_i(t_i) - C_j(t_j) + 2\pi\nu_0\left(\delta\ell_{ij}(t_i) + \Delta\ell_{ij}(t_i)\right) \\ &\quad + n^s_{ij}(t_j) - \hat{r}_{ij}(t_i)\cdot\left(\mathbf{n}^a_{ij}(t_j) - \mathbf{n}^a_{ji}(t_i)\right). \qquad (22)\end{aligned}$$

Here t_i is given implicitly by $t_i = t_j - \ell_{ij}(t_i)$ and ν_0 is the laser frequency. We have included the variations in the optical path length caused by gravitational waves, $\delta\ell_{ij}(t_i)$, and those caused by orbital effects, $\Delta\ell_{ij}(t_i)$. In what follows we will ignore the orbital contributions to the phase shift as they can be removed by high pass filtering. The subscripts on the noise sources identify the particular component that is responsible: C_i is the phase noise introduced by the laser on spacecraft i, n^s_{ij} denotes the shot noise in the photodetector on spacecraft j used to measure the phase of the signal from spacecraft i, and $\mathbf{n}^a_{ij}$ denotes the noise introduced by the accelerometers on spacecraft j that are mounted on the optical assembly that points toward spacecraft i.

The three LISA spacecraft will report six phase difference measurements which can then be used to construct a variety of interferometer outputs. The simplest are the three Michelson signals that can be formed by choosing one of the spacecraft as the vertex and using the other two as end-stations. The Michelson signal extracted from vertex 1 has the form

$$S_1(t) = \Phi_{12}(t_2) + \Phi_{21}(t) - \Phi_{13}(t_3) - \Phi_{31}(t)\,, \qquad (23)$$

where t_2 and t_3 are given implicitly by

$$\begin{aligned} t_2 &= t - \ell_{21}(t_2) \\ t_3 &= t - \ell_{31}(t_3)\,. \end{aligned} \qquad (24)$$

Unfortunately, the Michelson signals will be swamped by laser phase noise, so a more complicated virtual interferometer signal has to be used. The X variables are a set of three Michelson-like signals that cancel the laser phase noise [9]. The X signal extracted from vertex 1 has the form

$$\begin{aligned} X_1(t) &= \Phi_{12}(t_2) - \Phi_{12}(t_1 + t_2 - t) \\ &+ \Phi_{21}(t) - \Phi_{21}(t_1) \\ &- \Phi_{13}(t_3) + \Phi_{13}(t_1 + t_3 - t) \\ &- \Phi_{31}(t) + \Phi_{31}(t_1)\,. \end{aligned} \qquad (25)$$

The times t_1, t_2 and t_3 are defined implicitly:

$$\begin{aligned} t_1 &= t - \ell_{12}(t_1) - \ell_{21}(t_2) \\ &= t - \ell_{13}(t_1) - \ell_{31}(t_3) \\ t_2 &= t - \ell_{21}(t_2) \\ t_3 &= t - \ell_{31}(t_3)\,. \end{aligned} \qquad (26)$$

Given a gravitational wave signal $\mathbf{h}(q, \widehat{\Omega})$, a model of the instrument noise, and a description of the interferometer's orbit, we can use equations (15), (22) and (25) to calculate the detector response. Since the entire calculation is performed in Barycentric coordinates, the time t that appears in (23) and (25) is not the time τ measured by the clock on spacecraft 1. They are related by the standard time dilation formula $d\tau = dt\sqrt{1 - v_1^2(t)}$. However, since we only need to work to leading order in v, there is no need to distinguish between t and τ.

Our expression for the LISA response is much more complicated than the previous approximate descriptions. The time-variation of the optical path lengths is the main cause of the difficulty. It is responsible for the implicit relations that riddle the calculation. The path length variations have three main causes - intrinsic, tidal, and pointing. The intrinsic variations are part and parcel of the cartwheel orbit, which only keeps the distance between the spacecraft constant to leading order in the orbital eccentricity. The tidal variations are caused by the gravitational pull of other solar system bodies, most notably the Earth and Jupiter. The pointing corrections are a relativistic effect caused by the finite propagation speed of the lasers, which means that the spacecraft move between transmission and reception. The latter effect can be separated from the others:

$$\ell_{ij}(t_i) = L_{ij}(t_i)\left[1 + \hat{r}_{ij}(t_i)\cdot\mathbf{v}_j(t_i) + \mathcal{O}(v^2)\right] \qquad (27)$$

where $\mathbf{v}_j$ is the velocity of spacecraft j and

$$L_{ij}(t_i) = |\mathbf{x}_j(t_i) - \mathbf{x}_i(t_i)|. \qquad (28)$$

Ignoring tidal distortions and working to second order in the orbital eccentricity e, the orbits described in the Appendix yield

$$L_{12}(t) = L\left(1 + \frac{e}{32}\left[15\sin(\alpha + \frac{\pi}{6}) - \cos(3\alpha)\right]\right), \qquad (29)$$

and similar, yet slightly different, expressions for $L_{13}(t)$ and $L_{23}(t)$. Here $\alpha(t) = 2\pi f_m t + \kappa$ is the orbital phase and $f_m = 1/\text{year}$ is the orbital frequency. The mean armlength L is related to the eccentricity e and semi-major axis a by $L = 2\sqrt{3}ae$. Setting $L = 5\times 10^9$ meters yields $e = 0.00965 \approx 10^{-2}$. The spacecraft have velocities of order $v \approx 2\pi f_m a \approx 10^{-4}$. Using these numbers we see that the lowest order intrinsic variation is far larger than the pointing variation. The tidal variations turn out to be comparable to the intrinsic variation[6] and therefore should not be ignored.

III. STATIC LIMIT

As a point of reference, we can apply our general method to a static, equal arm detector interacting with a monochromatic, plane-fronted gravitational wave propagating in the $\widehat{\Omega} = -\hat{n}$ direction with principle polarization axes $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{h}(f, \xi) = A_+ e^{2\pi i f(t + \hat{n}\cdot\mathbf{x})}\boldsymbol{\epsilon}^+ + A_\times e^{2\pi i f(t+\hat{n}\cdot\mathbf{x})}\boldsymbol{\epsilon}^\times, \qquad (30)$$

where A_+ and $A_\times$ are complex constants and

$$\begin{aligned}\boldsymbol{\epsilon}^+ &= \hat{p}\otimes\hat{p} - \hat{q}\otimes\hat{q}, \\ \boldsymbol{\epsilon}^\times &= \hat{p}\otimes\hat{q} + \hat{q}\otimes\hat{p}. \end{aligned} \tag{31}$$

Defining the polarization angle $\psi = -\arctan(\hat{v}\cdot\mathbf{p}/\hat{u}\cdot\mathbf{p})$ we have

$$\begin{aligned}\boldsymbol{\epsilon}^+ &= \cos 2\psi\,\mathbf{e}^+ - \sin 2\psi\,\mathbf{e}^\times, \\ \boldsymbol{\epsilon}^\times &= \sin 2\psi\,\mathbf{e}^+ + \cos 2\psi\,\mathbf{e}^\times. \end{aligned} \tag{32}$$

Thus, in terms of the general decomposition (1) we have

$$\begin{aligned} h_+(t) &= (A_+\cos 2\psi + A_\times \sin 2\psi)\, e^{2\pi i f(t+\hat{n}\cdot\mathbf{x})}, \\ h_\times(t) &= (A_\times\cos 2\psi - A_+ \sin 2\psi)\, e^{2\pi i f(t+\hat{n}\cdot\mathbf{x})}. \end{aligned} \tag{33}$$

The signal portion of the X variable defined in (25) reduces to

$$\begin{aligned} X_1^s(t) = 2\pi\nu_0\Big(&\delta\ell_{12}(t-2L) - \delta\ell_{12}(t-4L) \\ &+\delta\ell_{21}(t-L) - \delta\ell_{21}(t-3L) - \delta\ell_{13}(t-2L) \\ &+\delta\ell_{13}(t-4L) - \delta\ell_{31}(t-L) + \delta\ell_{31}(t-3L)\Big) \end{aligned} \tag{34}$$

In terms of the strain $x_1(t) = X_1^s(t)/(2\pi L\nu_0)$ we have

$$x_1(t) = \mathbf{D}(\hat{n}, f) : \mathbf{h}(f,\xi)\sin^2(f/f_*) \tag{35}$$

where

$$\begin{aligned}\mathbf{D}(\hat{n}, f) &= \frac{1}{2}\left((\hat{r}_{12}\otimes\hat{r}_{12})\,\mathcal{T}(\hat{r}_{12}\cdot\hat{n}, f)\right. \\ &\quad \left. - (\hat{r}_{13}\otimes\hat{r}_{13})\,\mathcal{T}(\hat{r}_{13}\cdot\hat{n}, f)\right) \end{aligned} \tag{36}$$

and

$$\begin{aligned}\mathcal{T}(s,f) = \frac{1}{2}\Bigg[&\mathrm{sinc}\left(\frac{f(1+s)}{2f_*}\right)\exp\left(-i\frac{f}{2f_*}(3-s)\right) \\ &+ \mathrm{sinc}\left(\frac{f(1-s)}{2f_*}\right)\exp\left(-i\frac{f}{2f_*}(1-s)\right)\Bigg]. \end{aligned} \tag{37}$$

Orienting the detector in the $x-y$ plane according to Figure 2 of Ref. [1], we have

$$\begin{aligned}\hat{r}_{12} &= \cos(\pi/12)\,\hat{x} + \sin(\pi/12)\,\hat{y} \\ \hat{r}_{13} &= \cos(5\pi/12)\,\hat{x} + \sin(5\pi/12)\,\hat{y}. \end{aligned} \tag{38}$$

Combing these expression with equations (2) and (3) yields

$$\begin{aligned}(\hat{r}_{12}\otimes\hat{r}_{12}) : \mathbf{e}^+ &= \frac{1}{2}\left((1+\cos^2\theta)\sin(2\phi+\pi/3) - \sin^2\theta\right) \\ (\hat{r}_{12}\otimes\hat{r}_{12}) : \mathbf{e}^\times &= \cos\theta\,\sin(2\phi-\pi/6) \\ (\hat{r}_{13}\otimes\hat{r}_{13}) : \mathbf{e}^+ &= \frac{1}{2}\left((1+\cos^2\theta)\sin(2\phi-\pi/3) - \sin^2\theta\right) \\ (\hat{r}_{13}\otimes\hat{r}_{13}) : \mathbf{e}^\times &= -\cos\theta\,\sin(2\phi+\pi/6). \end{aligned} \tag{39}$$

The above collection of equations, (35) through (39), fully define the detector response in the static limit. The X variable response $x_1(t)$ is related to the Michelson response $s_1(t)$ by

$$s_1(t) = x_1(t)\sin^{-2}(f/f_*) = \mathbf{D}(\hat{n}, f) : \mathbf{h}(f,\xi). \tag{40}$$

Our expression (40) for $s_1(t)$ agrees with that quoted in Ref.[4]. In the low frequency limit, $f \ll f_*$, the transfer function reduces to unity, $\mathcal{T} = 1$, and

$$s_1(t) = \Big(A_+F^+(\theta,\phi,\psi) + A_\times F^\times(\theta,\phi,\psi)\Big)e^{2\pi i f t}. \tag{41}$$

The beam pattern factors

$$\begin{aligned}F^+(\theta,\phi,\psi) &= \frac{1}{2}(\hat{r}_{12}\otimes\hat{r}_{12} - \hat{r}_{13}\otimes\hat{r}_{13}) : \boldsymbol{\epsilon}^+ \\ F^\times(\theta,\phi,\psi) &= \frac{1}{2}(\hat{r}_{12}\otimes\hat{r}_{12} - \hat{r}_{13}\otimes\hat{r}_{13}) : \boldsymbol{\epsilon}^\times \end{aligned} \tag{42}$$

take their familiar form:[10]

$$\begin{aligned}F^+ &= \frac{\sqrt{3}}{2}\Big(\frac{1}{2}(1+\cos^2\theta)\cos 2\phi\cos 2\psi \\ &\qquad - \cos\theta\sin 2\phi\sin 2\psi\Big) \\ F^\times &= \frac{\sqrt{3}}{2}\Big(\frac{1}{2}(1+\cos^2\theta)\cos 2\phi\sin 2\psi \\ &\qquad + \cos\theta\sin 2\phi\cos 2\psi\Big). \end{aligned} \tag{43}$$

Notice that the overall factor of $\sqrt{3}/2$ compared to a detector with 90^o arms come out naturally in our calculation.

IV. LOW FREQUENCY LIMIT

As a further point of reference, we can apply our general result to the low frequency limit considered in Ref.[1]. The low frequency limit is defined by the condition $f \ll f_*$, where $f_* \equiv 1/(2\pi L)$ is the typical transfer frequency along each arm. A LISA mission with $L = 5\times 10^9$ meter arms has a transfer frequency of $f_* = 0.00954 \approx 10^{-2}$ Hz. The motion of the LISA constellation is included to leading order in the eccentricity, and the gravitational wave is taken to be monochromatic, plane-fronted and propagating in the $\widehat{\Omega} = -\hat{n}$ direction:

$$\mathbf{h}(t,\mathbf{x}) = A_+\boldsymbol{\epsilon}^+\cos(2\pi f(t+\hat{n}\cdot\mathbf{x})) + A_\times\boldsymbol{\epsilon}^\times\sin(2\pi f(t+\hat{n}\cdot\mathbf{x})). \tag{44}$$

The two orthogonal polarizations have constant real amplitudes A_+ and $A_\times$. The basic Michelson signal considered by Cutler[1] takes the form

$$\begin{aligned}s_1(t) &= \frac{\delta\ell_{12}(t-2L) + \delta\ell_{21}(t-L)}{2L} \\ &\quad - \frac{\delta\ell_{13}(t-2L) + \delta\ell_{31}(t-L)}{2L}. \end{aligned} \tag{45}$$

This expression ignores the time variation of the armlengths due to higher order terms in the orbital eccentricity or perturbations from other solar system bodies.

Using (15) we find

$$\begin{aligned} s_1(t) &= F^+(t)A_+\cos(2\pi f[t+\hat{n}\cdot\mathbf{x}_1(t)]) \\ &+ F^\times(t)A_\times\sin(2\pi f[t+\hat{n}\cdot\mathbf{x}_1(t)]), \end{aligned} \quad (46)$$

where

$$\begin{aligned} F^+(t) &= \frac{1}{2}\left(\cos 2\psi\, D^+(t) - \sin 2\psi\, D^\times(t)\right) \\ F^\times(t) &= \frac{1}{2}\left(\sin 2\psi\, D^+(t) + \cos 2\psi\, D^\times(t)\right), \end{aligned} \quad (47)$$

and

$$\begin{aligned} D^+(t) &\equiv (\hat{r}_{12}(t)\otimes\hat{r}_{12}(t) - \hat{r}_{13}(t)\otimes\hat{r}_{13}(t)) : \mathbf{e}^+ \\ D^\times(t) &\equiv (\hat{r}_{12}(t)\otimes\hat{r}_{12}(t) - \hat{r}_{13}(t)\otimes\hat{r}_{13}(t)) : \mathbf{e}^\times . \end{aligned} \quad (48)$$

The expression for the strain in the detector can be rearranged using double angle identities to read:

$$s_1(t) = A(t)\cos\left[2\pi f t + \phi_D(t) + \phi_P(t)\right] . \quad (49)$$

The amplitude modulation $A(t)$, frequency modulation $\phi_D(t)$ and phase modulation $\phi_P(t)$ are given by

$$A(t) = \left[(A_+F^+(t))^2 + (A_\times F^\times(t))^2\right]^{1/2} \quad (50)$$

$$\begin{aligned} \phi_D(t) &= 2\pi f\hat{n}\cdot\mathbf{x}_1(t) \\ &= 2\pi f a\sin\theta\cos(\alpha-\phi) \end{aligned} \quad (51)$$

$$\phi_P(t) = -\arctan\left(\frac{A_\times F^\times(t)}{A_+F^+(t)}\right) . \quad (52)$$

Using the orbits described in the Appendix, the coordinates of each spacecraft are given to leading order in the eccentricity by

$$\begin{aligned} x &= a\cos(\alpha) + ae\left(\sin\alpha\cos\alpha\sin\beta - (1+\sin^2\alpha)\cos\beta\right) \\ y &= a\sin(\alpha) + ae\left(\sin\alpha\cos\alpha\cos\beta - (1+\cos^2\alpha)\sin\beta\right) \\ z &= -\sqrt{3}ae\cos(\alpha-\beta), \end{aligned} \quad (53)$$

where $\alpha = 2\pi f_m t + \kappa$ is the phase of the guiding center and $\beta = 2n\pi/3 + \lambda$ is the relative phase of each spacecraft in the constellation ($n = 0,1,2$). The unit vectors $\hat{r}_{ij}(t)$ can be derived from the coordinates given in (53). Putting this all together yields

$$\begin{aligned} D^+(t) &= \frac{\sqrt{3}}{64}\Big[-36\sin^2\theta\sin(2\alpha(t)-2\lambda) \\ &+ (3+\cos 2\theta)\Big(\cos 2\phi(9\sin 2\lambda - \sin(4\alpha(t)-2\lambda)) \\ &\quad +\sin 2\phi(\cos(4\alpha(t)-2\lambda) - 9\cos 2\lambda)\Big) \\ &- 4\sqrt{3}\sin 2\theta\Big(\sin(3\alpha(t)-2\lambda-\phi) \\ &\quad -3\sin(\alpha(t)-2\lambda+\phi)\Big)\Big] \end{aligned} \quad (54)$$

and

$$\begin{aligned} D^\times(t) &= \frac{1}{16}\Big[\sqrt{3}\cos\theta\Big(9\cos(2\lambda-2\phi) \\ &\quad -\cos(4\alpha(t)-2\lambda-2\phi)\Big) \\ &- 6\sin\theta\Big(\cos(3\alpha(t)-2\lambda-\phi) \\ &\quad +3\cos(\alpha(t)-2\lambda+\phi)\Big)\Big] \end{aligned} \quad (55)$$

Finally, for circular Newtonian binaries, the polarization angle ψ can be related to the angular momentum orientation $\hat{L}\to(\theta_L,\phi_L)$ by

$$\begin{aligned} \tan\psi &= -\frac{\hat{v}\cdot\mathbf{p}}{\hat{u}\cdot\mathbf{p}} = \frac{\hat{L}\cdot\hat{u}}{\hat{L}\cdot\hat{v}} \\ &= \frac{\cos\theta\cos(\phi-\phi_L)\sin\theta_L - \cos\theta_L\sin\theta}{\sin\theta_L\sin(\phi-\phi_L)}, \end{aligned} \quad (56)$$

where we have used

$$\mathbf{p} = \hat{n}\times\hat{L}. \quad (57)$$

The parameters κ and λ define the initial location and orientation of the LISA constellation. They are related to the quantities $\bar{\phi}_0$ and α_0 in Cutler's[1] equations (3.3) and (3.6) according to $\kappa = \bar{\phi}_0$ and $\lambda = 3\pi/4 + \bar{\phi}_0 - \alpha_0$. Our compact expression (49) for the low frequency limit agrees with Cutler's[1] result, but the agreement is by no means obvious. The equality can be established using a computer algebra program or by direct numerical evaluation.

V. SPECTRAL NOISE

The variation in the optical path length will be reflected in the noise transfer functions. For example, the noise $n^s_{21}(t)$ enters into the X variable as

$$N(t) = n^s_{21}(t) - n^s_{21}(t_1) . \quad (58)$$

Writing $n^s_{21}(t) = n(t)$ and assuming that the armlengths are fixed, $\ell_{31}(t_1) = \ell_{31}(t_3) = L$, yields the standard result

$$N(f) = n(f)\left(1 - e^{2if/f_*}\right), \quad (59)$$

and

$$S_N(f) = 4\sin^2\left(\frac{f}{f_*}\right)S_n(f). \quad (60)$$

Here $S_n(f)$ is the noise spectral density in the photodetector and we have used the assumption

$$< n(f)n^*(f') >= \delta(f-f')S_n(f), \quad (61)$$

where the brackets <> denote an ensemble average. The situation is much more complicated when the armlengths vary since

$$N(f) = n(f) - \frac{1}{2\pi}\int n(t_1(t))e^{2\pi i f t}dt . \quad (62)$$

Because $\ell(t)$ varies with a one year period, the transfer function will develop sidebands at $f \pm n f_m$ where n takes integer values. Working to zeroth order in v and lowest order in e we have

$$t_1(t) \simeq t - 2L_{12}(t)\,. \tag{63}$$

Using the expression (29) for $L_{12}(t)$ and the expansion

$$e^{ix\sin(2\pi f_m t)} = \sum_{k=-\infty}^{\infty} J_k(x) e^{2\pi i f_m k t}, \tag{64}$$

where J_k is a Bessel function of the first kind of order k, allows us to write

$$N(f) = n(f) - \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} n(f+(j+3k)f_m)e^{2if/f_*}$$
$$\times e^{2if_m(j+3k)/f_*} e^{i\pi(j-3k)/6} J_j\left(\frac{15ef}{16f_*}\right) J_k\left(\frac{ef}{16f_*}\right) \tag{65}$$

The dependence on the Bessel functions tells us that the sidebands only become significant for frequencies approaching $f_*/e \approx 1$ Hz. Below the transfer frequency $f_* \sim 10$ mHz it is safe to ignore the time variation of the arm lengths in calculations of the noise transfer functions.

VI. DISCUSSION

We have shown that our general expression for the LISA response function reproduces the standard static and low frequency limits. However, we have said little about how the general result should be used. Given a specific model for the orbit, such as the simple Keplerian model described in the Appendix, it is possible to solve the implicit relations for the detector orientation, arm lengths and emission times, as we did in equations (27), (29) and (63). These can then be used to give explicit expressions for the Michelson or X variables. We did not quote these expressions as they are very large and not very informative. Ultimately, any application that requires the full LISA response function is likely to be numerical. It is a simple matter to write a computer program that returns the LISA response function using equations (25), (22) and (15). If one is only interested in sources with frequencies below ~ 5 mHz, the low frequency approximation (49) will suffice, but for accurate astrophysical parameter estimation above 5 mHz, the full LISA response function has to be used.

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Appendix: Keplerian Spacecraft Orbits

For a constellation of spacecraft in individual Keplerian orbits with an inclination of $i = \sqrt{3}e$ the coordinates of each spacecraft are given by the expressions

$$\begin{aligned} x &= r\left(\cos(\sqrt{3}e)\cos\beta\cos\gamma - \sin\beta\sin\gamma\right) \\ y &= r\left(\cos(\sqrt{3}e)\sin\beta\cos\gamma + \cos\beta\sin\gamma\right) \\ z &= -r\sin(\sqrt{3}e)\cos\gamma\,. \end{aligned} \tag{66}$$

where $\beta = 2n\pi/3 + \lambda$ ($n = 0,1,2$) is the relative orbital phase of each spacecraft in the constellation, γ is the ecliptic azimuthal angle, and r is the standard Keplerian radius

$$r = \frac{a(1-e^2)}{1+e\cos\gamma}\,. \tag{67}$$

Here a is the semi-major axis of the guiding center and has an approximate value of one AU.

To get the above coordinates as a function of time we first note that the azimuthal angle is related to the eccentric anomaly, ψ, by

$$\tan\left(\frac{\gamma}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{\psi}{2}\right)\,, \tag{68}$$

and the eccentric anomaly is related to the orbital phase $\alpha(t) = 2\pi f_m t + \kappa$ through

$$\alpha - \beta = \psi - e \sin \psi \,. \tag{69}$$

For small eccentricities we can expand equations (68) and (69) in a power series in e to arrive at

$$\gamma = (\alpha-\beta)+2e\sin(\alpha-\beta)+\frac{5}{2}e^2\sin(\alpha-\beta)\cos(\alpha-\beta)+\ldots \tag{70}$$

Substituting this series into equation (66) and keeping terms only up to order e gives us

$$\begin{aligned}
x &= a\cos(\alpha) + ae\left(\sin\alpha\cos\alpha\sin\beta - (1+\sin^2\alpha)\cos\beta\right) \\
y &= a\sin(\alpha) + ae\left(\sin\alpha\cos\alpha\cos\beta - (1+\cos^2\alpha)\sin\beta\right) \\
z &= -\sqrt{3}ae\cos(\alpha-\beta)\,.
\end{aligned} \tag{71}$$

These are the desired coordinates of each spacecraft as a function of time. Notice that by keeping only linear terms in the eccentricity we are neglecting the variation in the optical path length. The path length will change due to the Keplerian orbits, but these effects enter at $\mathcal{O}(e^2)$ and above.